THE CLASSIFICATION PROBLEM FOR FREE ERGODIC ACTIONS

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ABSTRACT. These are preliminary notes for the talk I will give in the "Sexto Coloquiot Uruguayo de Matemática". The material is based on joint work with Martino Lupini.

1. MOTIVATION AND STATEMENT OF THE MAIN RESULT

Throughout this talk, we fix an infinite countable group Γ, and a standard probability space (X, μ) (to fix ideas, take $[0,1]$ with the Lebesgue measure). We are interested in studying actions of Γ on (X, μ) by measure-preserving transformations.

Definition 1.1. An action $\Gamma \curvearrowright (X, \mu)$ is said to be:

(1) *free* if $\mu(\{x \in X : \gamma \cdot x = x\}) = 0$ for all $\gamma \in \Gamma \setminus \{1\}$.
(2) *ergodic* if whenever E is Γ-invariant, then $\mu(E) = 0, 1$.

Remark 1.2. If Γ is infinite and acts freely on (X, μ), then X can have no atoms. Indeed, if $x \in X$ has positive measure, then each of its translates has the same measure. The translates are all distinct by freeness, so the orbit of x has infinite measure.

This talk focuses on the classification of free, ergodic actions, up to the following equivalence relations:

Definition 1.3. Let $\theta, \kappa \colon \Gamma \curvearrowright (X, \mu)$ be actions. We say that

(1) θ is *conjugate* to κ, written $\theta \cong \kappa$, if there exists $\varphi \in \mathrm{Aut}(X, \mu)$ such that $\varphi \circ \theta_\gamma = \kappa_\gamma \circ \varphi$ almost everywhere for all $\gamma \in \Gamma$.
(2) θ is *orbit equivalent* to κ, written $\theta \sim_{\mathrm{OE}} \kappa$, if there exists $\varphi \in \mathrm{Aut}(X, \mu)$ such that $\varphi(\theta(\Gamma) \cdot x) = \kappa(\Gamma) \cdot \varphi(x)$ for almost every $x \in X$.

Conjugacy is rather strong: for $\mathbb{Z}$-actions, θ is rarely conjugate to θ^{-1}. However, θ is always orbit equivalent to θ^{-1}. In fact, orbit equivalence admits a nice operator-algebraic characterization:

Theorem 1.4. $\theta \sim_{\mathrm{OE}} \kappa$ if and only if there is an isomorphism $\psi \colon L^\infty(X, \mu) \rtimes_\theta \Gamma \to L^\infty(X, \mu) \rtimes_\kappa \Gamma$ satisfying $\psi(L^\infty(X, \mu)) = L^\infty(X, \mu)$.

The following questions of Halmos motivate our work:

Question 1.5 (Halmos; 1956)**.** Is there a *method* to determine whether two given (free, ergodic) actions of Γ are conjugate/orbit equivalent?

By Halmos' own admission, this is a vague question. A formal interpretation can be given in the context of Borel complexity theory:

Question 1.6 (Kechris; 2006)**.** Are the relations of conjugacy or orbit equivalence of (free, ergodic) actions of Γ Borel?

We need to specify what the meaning of "Borel" is in the question above. Suppose that E is a complete metric space with a notion of equivalence $\sim$ on it. We can regard $\sim$ as a subset of $E \times E$, namely

$$\sim \, = \{(e, f) \in E \times E \colon e \sim f\}.$$

One says that $\sim$ is Borel, if it is a Borel subset of $E \times E$ (with the product σ-algebra). The relation is Borel precisely if there exists an explicit uniform procedure that, given two elements in E, runs for countably many

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steps, at each step testing membership in some given open sets, and at the end decides whether the elements are equivalent or not.

For the above to apply to Γ-actions, we need to explain how we regard (free, ergodic) actions as a complete metric space. First, observe that $\mathrm{Aut}(X,\mu)$ is a complete metric space with the weak topology (convergence in measure). We endow $\mathrm{Aut}(X,\mu)^\Gamma$ with the product topology, and $\mathrm{Act}_\Gamma(X,\mu)$ is a closed subspace of it. We can furthermore restrict to the spaces of free and ergodic actions.

One expects conjugacy to be a much more complicated equivalence relation. It is bad already for $\mathbb{Z}$:

Theorem 1.7 (Foreman-Rudolph-Weiss; Annals 2010)**.** The relation of conjugacy of free, ergodic automorphisms is not Borel.

Things are better for orbit equivalence:

Theorem 1.8 (Dye; AJM 1959, Ornstein-Weiss; Memoirs 1980)**.** Any two free, ergodic $\mathbb{Z}$-actions are orbit equivalent. More generally, any two free, ergodic actions of an amenable group are orbit equivalent.

There are also "models": the Bernoulli shift $\beta\colon \Gamma \curvearrowright [0,1]^\Gamma$.

Thus, for orbit equivalence, Halmos' question is only interesting for non-amenable groups. Some partial results:

- Connes-Weiss: if Γ doesn't have property (T), then it admits at least two non-OE free, ergodic actions.
- Hjorth [TAMS, 2005] if Γ has property (T), then it admits uncountably many actions.
- Gaboriau-Popa [JAMS, 2005]: $\mathbb{F}_n$, for $1 < n < \infty$, admits uncountably many actions.
- Ioana [Inv., 2007]: if Γ contains $\mathbb{F}_2$, then it admits uncountably many actions.
- Epstein [unpublished, 2007]: any nonamenable group admits uncountably many actions.
- Epstein-Törnquist [unpublished, 2011]: if $\mathbb{F}_2 \triangleleft \Gamma$, then OE of Γ-actions is not Borel.

Of course, having uncountably many OE-classes does not imply non-Borelness.

Here, we answer Halmos' questions for nonamenable groups, also generalizing the above mentioned results:

Theorem 1.9 (G.-Lupini)**.** If Γ is nonamenable, then conjugacy and OE of free, ergodic actions are not Borel.

Together with the results of Dye and Ornstein-Weiss, we have the following dychotomy:

Corollary 1.10. Let Γ be an infinite countable group.

(1) If Γ is amenable, then any two free, ergodic actions of Γ are OE.
(2) If Γ is nonamenable, then conjugacy and OE of free, ergodic actions is not Borel.

One concrete application of results of this nature is to rule out certain classification theorems. For example:

Theorem 1.11 (Bowen; Annals 2010)**.** Bernoulli shifts of free groups are classified by their entropy.

Can entropy classify all free, ergodic actions? It can't, because otherwise the relation would be Borel.

We can't rule out classification by groups, since OE is equivalent to isomorphism of the full groups.

2. Sketch of the proof when Γ contains a free group

Definition 2.1. Let $\Gamma \curvearrowright^\theta (X,\mu)$ be an action and let $\Delta \leq \Lambda \leq \Gamma$ be subgroups.

(1) A θ-*cocycle* is a function $u\colon \Gamma \to \mathcal{U}(L^\infty(X,\mu))$ such that $u_{\gamma\rho} = u_\gamma \theta_\gamma(u_\rho)$ for all $\gamma,\rho\in\Gamma$.
(2) A θ-cocycle u is said to be Δ-*invariant* if $u_\delta = 1$ and $\theta_\delta(u_\gamma) = u_\gamma$ for all $\delta\in\Delta$ and all $\gamma\in\Gamma$.
(3) A θ-cocycle u is said to be a Λ-*relative weak coboundary* if there exists $z\in\mathcal{U}(L^\infty(X,\mu))$ such that $u_\lambda = z\theta_\lambda(z^*)$ up to scalars, for all $\lambda\in\Lambda$.
(4) The Δ-invariant Λ-relative weak 1-cohomology group of θ is the quotient $H^1_{:\Delta,\Lambda,w}(\theta)$ of Δ-invariant cocycles by the coboundaries.

$H^1_{:\Delta,\Lambda,w}(\theta)$ has a natural group structure, and it is a conjugacy invariant.

Remark 2.2. When $\Delta = \{e\}$ and $\Lambda = \Gamma$, we recover the usual definitions. When Δ is a normal subgroup of $\Lambda = \Gamma$, this is essentially the cohomology of the quotient action.

Definition 2.3. A triple $\Delta \leq \Lambda \leq \Gamma$ is said to have *property (T)* if any unitary representation of Γ with almost invariant Δ-invariant vectors, has a Λ-invariant vector.

For example, $\{1\} \leq \Gamma \leq \Gamma$ has propery (T) if and only if Γ has property (T). For $\Delta = \{1\}$, we obtain the relative property (T). Also, if Δ is a normal subgroup of Λ and Λ/Δ has property (T), then so does the triple.

Adapting methods of Popa, we prove a general superrigidity theorem for malleable actions of groups with nested subgroups with property (T). In our context, we will use it as follows:

Theorem 2.4 (G.-Lupini)**.** Let $\Delta \leq \Lambda \leq \Gamma$ have property (T), and suppose that Δ has infinite index in Γ. Let $\beta\colon \Gamma \curvearrowright L^\infty(X,\mu)^{\otimes \Gamma/\Delta} \otimes L^\infty(X,\mu)^{\otimes\Gamma}$ be the canonical shift. Then

$$H^1_{:\Delta,\Lambda,w}(\beta) = \{1\}.$$

Now, we explain how the construction in the proof of Theorem 1.9 works when Γ contains $\mathbb{F}_2$. We fix an inclusion $\mathbb{F}_2 \leq \mathrm{SL}_2(\mathbb{Z})$ and denote by $\rho\colon \mathbb{F}_2 \curvearrowright \mathbb{T}^2$ the induced action.

Notation 2.5. We (co)induce ρ to an action of Γ as follows. Set

$$Y = \{f\colon \Gamma \to \mathbb{T}^2\colon f(\gamma a) = \rho_{a^{-1}}(f(\gamma)) \text{ for all } \gamma \in \Gamma, a \in \mathbb{F}_2\} \subseteq (\mathbb{T}^2)^\Gamma,$$

endowed with the restriction ν of the product measure. Then (Y,ν) is an atomless standard probability space, and we define $\widehat{\rho}\colon \Gamma \curvearrowright (Y,\nu)$ by $\widehat{\rho}_{\gamma_0}(f)(\gamma_1) = f(\gamma_0^{-1}\gamma_1)$ for all $\gamma_0, \gamma_1 \in \Gamma$.

We will prove the following:

Theorem 2.6. There is a (Borel) assignment $A \mapsto \theta_A$ from countably infinite abelian groups to freem ergodic actions of Γ satisfying:

1. $A \cong A'$ if and only if $\theta_A \cong \theta_{A'}$, and
2. if $\mathcal{A}$ is a collection of abelian groups such that $\{\theta_A\colon A \in \mathcal{A}\}$ are pairwise not conjugate but are all OE, then $\mathcal{A}$ is countable.

With this, it follows that there is a countable-to-one (Borel) reduction from isomorphism of abelian groups to OE of free, ergodic actions. This is known to imply the result [Epstein-Törnquist; Montalbán].

Proof. (of the Theorem above). Fix a normal subgroup Δ of $\mathbb{F}_2$ whose quotient is infinite and has property (T). It follows that $\Delta \leq \mathbb{F}_2 \leq \Gamma$ has property (T). Let A be a countably infintie abelian group, and let G be its Pontryagin dual. Then G is compact and second countable. Set $M = L^\infty(G)$. Consider the following actions:

$$\Gamma \curvearrowright^{\beta} M^{\otimes \Gamma/\Delta} \otimes M^{\otimes\Gamma} \curvearrowleft^{\alpha} G,$$

where β is the tensor product of the Bernoulli shifts, and α is the tensor product of $\mathtt{Lt}^{\otimes\Gamma/\Lambda}$ and $\mathrm{id}_{M^{\otimes\Gamma}}$. (Observe that α has the Rokhlin property.)

Let M_A denote the fixed point algebra of α; then $M_A \cong L^\infty(X_A,\mu_A)$ for some standard atomless probability space (X_A,μ_A). Since α and β commute, it follows that β induces an action $\beta_A\colon \Gamma \curvearrowright M_A$.

Define $\theta_A = \beta_A \otimes \widehat{\rho}$, which is an action of Γ on the atomless standard probability space. It is free because it has the free action β_A as a factor. Using that Γ/Δ is infinite, and properties of $\widehat{\rho}$, one can show that this action is ergodic.

Claim: *there is a group isomorphism* $H^1_{:\Delta,\mathbb{F}_2,w}(\theta_A|_{\mathbb{F}_2}) \cong A$. (This implies part (1) of the statement.) Let w be a Δ-invariant cocycle for $\theta_A|_{\mathbb{F}_2}$. Then w is also a Δ-invariant cocycle for $\beta|_{\mathbb{F}_2} \otimes \widehat{\rho}|_{\mathbb{F}_2}$. Moreover, $\widehat{\rho}|_{\mathbb{F}_2}$ is "closely related" to ρ (it's essentially an amplification of it). By superrigidity (Theorem 2.4), w is a coboundary for $\beta|_{\mathbb{F}_2} \otimes \widehat{\rho}|_{\mathbb{F}_2}$, which means that there exists a unitary $v \in \mathcal{U}(M^{\otimes\Gamma/\Delta})$ implementing the triviality. Using ergodicity, one shows that there exists a character $\chi_w \in \widehat{G} \cong A$ satisfying

$$\alpha_g(v) = \chi_w(g)v$$

for all $g \in G$. The rest of the proof consists in showing that the assignment $[w] \mapsto \chi_w$ is a group isomorphism. We omit the details. This already shows that conjugacy is not Borel.

Part (2) of the statement is also involved, and exploits the rigidity properties of ρ. □

The notion of a triple of groups with property (T) is new, and it is arguably the main novelty since it gives us access to cocycle superrigidity even for actions of free groups, which are known not to contain subgroups with the relative property (T). Perhaps for this reason, all other proofs (even for free groups) had to avoid the use of superrigidity techniques, and hence could only produce uncountably many actions (very concretely, they are indexed by the tracial values of projections in the hyperfinite II_1-factor, which is $[0,1]$).

We have shown the basic ideas for the case that Γ contains $\mathbb{F}_2$, but this is of course not enough. The von Neumann problem asked whether every nonamenable group contains $\mathbb{F}_2$; this problem was open for many decades, until it was solved in the negative in the 80's. (In fact, there are nonamenable groups all of whose proper subgroups are finite!) However, it has very recently been shown by Gaboriau and Lyons that this problem has an affirmative "measurable" solution:

Theorem 2.7 (Gaboriau-Lyons; Inv. 2009)**.** If $\Gamma \curvearrowright (X,\mu)^\Gamma$ is the Bernoulli shift of a nonamenable group Γ, then there exists a free ergodic action $\mathbb{F}_2 \curvearrowright (X,\mu)^\Gamma$ such that $\mathbb{F}_2 \cdot x \subseteq \Gamma \cdot x$ for almost every $x \in X^\Gamma$.

This gives an embedding of the orbit equivalence relations. The right setting in which to apply these results seems to be that of étale groupoids: in this context, we develop a notion of property (T) for triples of groupoids and prove a superrigidity theorem analogous to Theorem 2.4. The coinduction process is done at the level of the full groups of the (transformation) groupoids. This approach requires significantly more work, and gives the result in full generality.

In fact, the more feneral framework allows us to prove the result in the much more general context of extensions of equivalence relations. This approach has the advantage of also allowing us to drop the assumption of discreteness in Theorem 1.9:

Theorem 2.8. Let G be a nonamenable, second countable locally compact unimodular group. Then conjugacy and OE of free, ergodic actions of G are not Borel.

The amenable case of OE was treated by Connes-Feldman-Weiss.

There are many interesting questions. Here are two:

Question 2.9. Is conjugacy for free, ergodic actions of infinite discrete groups not Borel?

For $\mathbb{Z}$ and for nonamenable groups, the answer is yes. It is open for general amenable groups. What about general locally compact unimodular groups?

Question 2.10. Is there an analog of Gaboriau-Lyons' measurable solution to the von Neumann problem for outer actions on the hyperfinite II_1-factor $\mathcal{R}$? Does the Bernoulli shift $\Gamma \curvearrowright \otimes_\Gamma \mathcal{R}$ "contain" an outer action of $\mathbb{F}_2$ on $\mathcal{R}$?

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