Publicaciones Matemáticas del Uruguay
Volumen XX, Xxxx XXXX, Páginas xxx–xxx
S 0797-1443

WAGNER AND PETERSEN ARE UNIFORMLY MOST-RELIABLE GRAPHS

PABLO ROMERO

Abstract. If we are given n nodes and m links, what is the most reliable network topology? Partial answers are offered in the literature. Here, we show that Wagner and Petersen graphs are uniformly most-reliable graphs, and a conjecture on the existence of new such graphs is provided.

1. Motivation

Extremal graph theory is inspirational for network design [9]. In the second book ever written in graph theory, Berge challenges the readers to find the graph with maximum connectivity among all graphs with a fixed number of nodes and links. Frank Harary provided not only a full answer, but also found connected graphs with minimum and maximum diameter [16]. Gustav Kirchhoff solved linear time-invariant resistive circuits, and as corollary he introduced the Matrix-Tree theorem, where he counts the number of spanning trees of a graph (i.e., the tree-number) using the determinant of a matrix [18]. This breakthrough in electrical systems launched the theory of trees, which provides the building blocks in communication design [16]. However, the corresponding extremal problem is not well understood: find the graph with a fixed number of nodes and links that maximizes the tree-number [24].

The previous problems are deterministic. In network reliability analysis, the goal is to determine the probability of correct operation of a system [15, 4]. In its most elementary setting, we are given a simple graph G with perfect nodes but random link failures with identical and independent probability ρ.

Even though network reliability is probabilistic in nature, there is a strong interplay with the previous deterministic problems. The motivation of this paper is to have a better understanding of the interplay between network reliability analysis and deterministic problems. The main contributions are the following:

(1) The interplay between uniformly most-reliable graphs and easy graphs is considered.
(2) Optimal augmentations of a cycle are produced, which lead us to Möbius graphs and Wagner graph in a special case.
(3) Wagner and Petersen graphs are formally proved to be uniformly most-reliable.
(4) A conjecture on uniformly most reliable graphs is here posed, inspired by Wagner graph and related works on the field.

This paper is organized as follows. Section 2 presents a formal definition of reliability polynomial, easy graphs and uniformly most-reliable graphs. Section 3 covers the body of related works on uniformly most-reliable graphs.

The practical value and potential application of uniformly most-reliable graphs in communication systems is discussed in Section 4. Augmentations arise as a natural approach once the network is already deployed. Assuming that most communication systems are 2-node connected, the analysis is focused on iterative augmentations of a cycle.

The result of these augmentations is Möbius graphs M_n. In Section 5 it is formally proved that Wagner graph M_4 is uniformly most-reliable. However, M_5 does not belong in the category, since Petersen graph is uniformly most-reliable.

Inspired by prior works in the field, it is conjectured that $(n, n+4)$ uniformly most-reliable graphs are obtained by successive elementary subdivisions of Wagner graph. Finally, Section 6 presents concluding remarks and trends for future work.

2. Uniformly Most-Reliable Graphs

We are given a simple graph $G = (V, E)$, with perfect nodes and unreliable links with failure probability ρ. The all-terminal reliability $R_G(\rho)$ measures the probability that the resulting random graph remains connected, and it is a polynomial in $\rho \in [0, 1]$. For convenience, in this paper we work with the unreliability polynomial $U_G(\rho) = 1 - R_G(\rho)$. Let us denote $p = |V|$ and $q = |E|$ the respective order and size of the graph G. Further, denote by $m_k(G)$, or simply m_k, the number of link-disconnecting sets with cardinality k, this is, the number of subsets $E' \subseteq E$ such that $|E'| = k$ and $G' = G - E'$ is disconnected. By sum-rule, the unreliability polynomial can be expressed as follows:

$$U_G(\rho) = \sum_{k=0}^{q} m_k \rho^k (1-\rho)^{q-k}. \tag{1}$$

Let us denote (p, q)-graph to the family of graphs with p nodes and q links. Clearly, if we consider a fixed ρ, there is at least one graph H that attains the minimum unreliability, i.e., $U_H(\rho) \leq U_G(\rho)$ for all (p, q) graph G. Further, if the previous condition holds for all $\rho \in [0, 1]$ and all (p, q)-graphs G, the graph H is uniformly most-reliable.

The determination of the unreliability polynomial is equivalent to finding the coefficients m_k for all k. Ball and Provan showed that this problem belongs to the $\mathcal{NP}$-Hard class [1]. However, they also proved that the determination of m_λ is feasible in polynomial time. On the other hand, Kirchhoff found the tree-number $\tau(G)$ efficiently, so the number $m_{q-p+1} = \binom{q}{q-p+1} - \tau(G)$ can be obtained in polynomial time [18]. Observe that $m_k = 0$ whenever $k < \lambda$, and $m_k = \binom{q}{k}$ if $k > q - p + 1$. Under these considerations, the unreliability polynomial can be rewritten [25]:

$$
\begin{aligned}
U_G(\rho) &= m_\lambda \rho^\lambda (1-\rho)^{q-\lambda} + \sum_{k=\lambda+1}^{q-p} m_k \rho^k (1-\rho)^{q-k} \\
&+ (\binom{q}{q-p+1} - \tau(G))\rho^{q-p+1}(1-\rho)^{p-1} \\
&+ \sum_{k=q-p+2}^{q} \binom{q}{k} \rho^k (1-\rho)^{q-k}.
\end{aligned} \tag{2}
$$

Expression (2) holds whenever $\lambda+1 \leq q-p$. Observe that in the remaining cases all the coefficients m_k are fully known beforehand, and the unreliability polynomial accepts a straightforward calculation [12].

Definition 1 (Level of difficulty)**.** *The level of difficulty of a graph G, denoted by $d(G)$, is the number of unknown coefficients from the unreliability polynomial:*

$$d(G) = (q-p) - (\lambda+1) + 1 = (q-p+1) - \lambda - 1. \tag{3}$$

If we are given a connected (p,q)-graph, then $q-p+1$ represents the difference between its size and the size of a spanning tree. The number $s = q-p+1$ is called the *co-rank* of a graph for algebraic reasons [5].

The invariant $d(G)$ measures *how difficult* is the problem of finding the unreliability polynomial. A valuable case occurs when $d(G) \leq 0$:

Definition 2 (Easy graph)**.** *A graph G is easy if $d(G) \leq 0$.*

The following result is just a corollary of the definition of easy graphs:

Corollary 1. *The coefficients m_k are fully known finding only $\tau(G)$ and m_λ, if and only if G is an easy graph.*

If we delete more than $s = q-p+1$ edges of a (p,q)-graph, the resulting subgraph is not connected. Therefore, $\lambda \leq s+1$. However, the reader can observe that the equality is achieved in trees and elementary cycles. In [12], it is proved that they are the only graphs where the equality $\lambda = s+1$ holds. In other terms, they belong to the class of easy graphs with the lowest level of difficulty $d = -2$.

Corollary 2. *All $(n, n+i)$ uniformly most-reliable graphs with $i < n/2$ have level of difficulty $d = i-2$.*

Proof. The maximum connectivity among $(n, n+i)$-graphs is $\lambda_{max} = \lfloor 2 + \frac{2i}{n} \rfloor = 2$. The co-rank is $s = (n+i) - n + 1 = i+1$. Therefore, the level of difficulty of all $(n, n+i)$ uniformly most-reliable graphs with $i < n$ is $d = (i+1) - 2 - 1 = i - 2$. □

A full characterization of easy graphs is already presented [12]. An exhaustive search among easy graphs reveals a simple way to prove that trees, cycles, Monma graphs and elementary subdivisions of K_4 and $K_{(3,3)}$ are uniformly most-reliable graphs for $i \in \{-1, 0, 1, 2, 3\}$ respectively.

3. Related Work

From inspection of Expression (1), we can see that if there exists some (p,q)-graph H such that $m_k(H) \leq m_k(G)$ for all k and all (p,q)-graph G, then H is uniformly most-reliable. The level of difficulty is a natural way of understanding the coefficient-based approach of uniformly most-reliable graphs. Curiously enough, this sufficient criterion is not known to be necessary. However, to the best of our knowledge, the search of uniformly most-reliable graphs rests on the minimization of all the coefficients m_k. This approach is promoted by the following result, which can be proved using elementary calculus [2]:

Proposition 1.

(i) *If there exists some integer k such that $m_i(H) = m_i(G)$ for all $i < k$ but $m_k(H) < m_k(G)$, then there exists $\rho_0 > 0$ such that $U_H(\rho) < U_G(\rho)$ for all $\rho \in (0, \rho_0)$.*

(ii) *If there exists some integer k such that $m_i(H) = m_i(G)$ for all $i > k$ but $m_k(H) < m_k(G)$, then there exists $\rho_1 < 1$ such that $U_H(\rho) < U_G(\rho)$ for all $\rho \in (\rho_1, 1)$.*

By definition, there are no disconnecting sets with lower cardinality than the link connectivity. Therefore, $m_i(G) = 0$ for all $i < \lambda$, and by Proposition 1-(i) uniformly most-reliably graphs must have the maximum link-connectivity λ. Furthermore, the number of disconnecting sets m_λ must be minimized. On the other hand, $m_i(G) = \binom{q}{i}$ for all $i > q - p + 1$, since trees are minimally connected with $q = p - 1$ links. The number of connected sets with $q - p + 1$ links is precisely the tree-number $\tau(G)$, so $m_{q-p+1}(G) = \binom{q}{q-p+1} - \tau(G)$. Using Proposition 1-(ii), the tree-number should be maximized. Prior observations directly connect this network design problem with extremal graph theory:

Corollary 3. *A uniformly most-reliable (p, q)-graph H must have the maximum tree-number $\tau(H)$, maximum connectivity $\lambda(H)$, and the minimum number of disconnecting sets $m_\lambda(H)$ among all (p, q)-graphs with maximum connectivity.*

For convenience we say that a (p, q)-graph, H, is t-optimal if $\tau(H) \geq \tau(G)$ for every (p, q) graph G. Briefly, Corollary 3 claims that uniformly most-reliable graphs must be t-optimal and max-λ min-m_λ, where λ denotes the edge connectivity.

Frank Harary found the maximum connectivity of a (p, q) graph. By handshaking, the average-degree of every (p, q)-graph is $\frac{2q}{p}$. If we denote $\delta(G)$ and $\lambda(G)$ the minimum degree and link-connectivity respectively, we immediately get that $\lambda(G) \leq \delta(G) \leq \lfloor \frac{2q}{p} \rfloor$. The candidate connectivity is $\lambda_{max} = \lfloor \frac{2q}{p} \rfloor$. It suffices to find a (p, q)-graph with connectivity λ_{max} whenever $p \geq q - 1$ (otherwise, the graph is not connected). The evidence is the following family of graphs [16]:

Definition 3 (Harary Graphs $H_{(n,k)}$)**.** *Let n and k be positive integers. Harary graph $H_{(n,k)}$ consists of n nodes $\{v_0, \ldots, v_{n-1}\}$ equally spaced around a circle, and the following links:*

- *If k is even, each vertex is adjacent to the $k/2$ nearest nodes in each direction.*
- *If k is odd and n is even, $H_{(n,k)}$ is $H_{(n,k-1)}$ with additional links $\{v_i, v_{i+\frac{n}{2}}\}$ for each $i = 0, \ldots, \frac{n}{2}$.*
- *If k and n are both odd, $H_{(n,k)}$ is $H_{(n,k-1)}$ with additional links $\{v_i, v_{i+\frac{n-1}{2}}\}$ for each $i = 0, \ldots, n-1$.*

We immediately check that Harary graphs have maximum connectivity $\lambda_{\max} = \lfloor \frac{2q}{p} \rfloor$, so, they are max-$\lambda$. The number of disconnecting sets should be minimized as well; max-λ graphs that minimize the disconnecting sets with λ nodes are called max-λ min-m_λ graphs. Prior works from Bauer et. al. fully characterize max-λ min-m_λ graphs [3]. A key idea is to observe that in max-λ graph, the number of disconnecting sets m_λ is at least the number of nodes with degree λ. If this bound is achieved, a max-λ min-m_λ graph is retrieved. For that purpose, they define generalized Harary graphs, which are just an augmentation of the original Harary graphs with random matchings (this is, edges with non-adjacent nodes). In that way, the number of nodes with degree λ is minimized, and the authors show that no other disconnecting sets with that size exists.

By Corollary 3, Bauer et. al. provide a family of graphs that contain all uniformly most-reliable graphs. Later works try to find uniformly $(p, p+i)$-most-reliable graphs for i small, by a simultaneous minimization of all the coefficients m_k. The cases $i = -1$ and $i = 0$ are trivial. Indeed, when $q = p - 1$ all the trees have the same reliability polynomial ρ^q, so they are uniformly most-reliable (the reliability is zero if the graph is not connected). When $i = 0$ we have $q = p$, and the elementary cycle C_p is t-optimal. All the other graphs with $p = q$ are not 2-connected, and by direct inspection we can see that C_p achieves the minimum coefficients m_k.

Perhaps the first non-trivial uniformly most-reliable graphs were found by Boesch et. al. in 1991 [7]. A new reading of Bauer et. al. construction lead them to find that Monma graphs are $(n, n+1)$ uniformly most-reliable graphs, whenever the number of nodes in each path differ by at most one. Interestingly enough, Clyde Monma et. al. used these graphs for the design of minimum cost two-node connected metric networks [22]. Figure 1 depicts Monma graphs. The reader is invited to find a combinatorial proof of Monma's t-optimality when the length of the paths differ by at most one in [13].

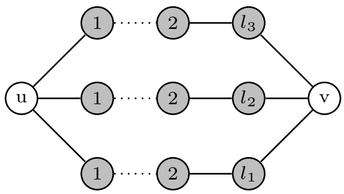

FIGURE 1. Monma graph $M_{(l_1+1, l_2+1, l_3+1)}$.

A more challenging problem is to find $(n, n+2)$ uniformly most-reliable graphs. Boesch et. al. minimize the four effective terms m_0, m_1, m_2 and m_3 from Expression (1). An $(n, n+2)$ max-λ min-m_λ graph already minimizes the first three terms. If in addition the tree-number is minimized all the coefficients are simultaneously minimized, and the result must be a uniformly most-reliable graph. The merit of the paper [7] is to adequately select the feasible graphs from Bauer et. al. that minimizes the tree-number. Observe that K_4 can be partitioned into three perfect matchings, PM_1, PM_2 and PM_3. The result is that we should insert $n-4$ points in the six links of K_4 in such a way that:

i the number of inserted nodes in all the links differ by at most one, and
ii if we insert the same number of nodes in two different matchings $PM_i \neq PM_j$, then the number of nodes in the four links from $PM_i \cup PM_j$ are identical.

The resulting $(n, n+2)$-graph defines, for every $n \geq 4$, a single graph up to isomorphism. The authors formally prove that the resulting graph is uniformly most-reliable $(n, n+2)$-graph. Furthermore, inspired by a previous research on t-optimality in multipartite graphs authored by Cheng [14], they conjecture that the shape of uniformly most-reliable $(n, n+3)$-graphs are elementary subdivisions of $K_{(3,3)}$. This conjecture is correct, and it was proved by Wang [26]. To the best of our knowledge, a full characterization of $(n, n+4)$-graphs is still open.

It is worth to remark that there are (p, q)-pairs where a uniformly most-reliable graphs does not exist [23]. The reader can consult [8] for a valuable survey on uniformly most-reliable graphs.

A full determination of t-optimal graphs for every (p, q)-pair is a related open problem. Indeed, a historical result credited by Leggett and Bedrosian asserts that t-optimal graphs must be almost regular, this is, the degrees do not differ by most than one [20]. Even though closed formulas are available for the tree-number of specific graphs, the progress on t-optimality is effective on special regularity conditions [14], almost-complete graphs or other special graphs with few links [24].

4. Practice and Augmentation

In this section we highlight potential application of uniformly most-reliable graphs, as an inspirational design tool. Real-life fiber-optics communication is already deployed in most countries, and a cost-effective solution is to add a single link in order to maximize the reliability. This is an augmentation problem: given a simple graph G, add a single link e such that the unreliability $U_{G \cup e}(\rho)$ is minimized.

We know from previous result on uniformly most-reliable graphs that an unreliability minimization over the whole compact set $p \in [0, 1]$ is not always feasible for all graphs. Since fiber-optics are highly-reliable systems the elementary unreliability ρ is small, and by Proposition 1, the unreliability is $U_G(\rho) \approx m_\lambda \rho^\lambda (1-\rho)^{q-\lambda}$, where λ is the link-connectivity. Furthermore, we know that real-life physical implementation of fiber-optics are 2-node connected, so, every pair of nodes are included in a ring.

As a point of departure, we assume we are given a ring $G = C_n$ with n even, and we study the augmentation problem in a step-by-step fashion. Specifically, we want to find the sequence of graphs $\{G^{(i)}\}_{i=0,\ldots,\lfloor \frac{n}{2} \rfloor}$ with $G^{(0)} = C_n$, such that the graph $G^{(i+1)} = G^{(i)} \cup \{e_{i+1}\}$ represents the best augmentation. In words, we consider iterative augmentations of the cycle. In network planning, this means that the operator decides to add a single link greedily, for instance, at different dates. In the following paragraphs we find the sequence $\{G^{(i)}\}_{i=0,\ldots,\lfloor \frac{n}{2} \rfloor}$.

By Handshaking Lemma, 3-regular graphs with n nodes must have $m = 3n/2$ links. Thus, if we add less than $\lfloor \frac{n}{2} \rfloor$ links to the cycle C_n there is at least one node v_i with degree $deg(v_i) = 2$, and $\lambda(G^{(i)}) = 2$ for all $i < \lfloor \frac{n}{2} \rfloor$. This means that we should minimize the coefficient m_2 in a step-by-step fashion.

The key is to observe that a selected class of disconnecting sets is subsequently partitioned into *two classes*, whenever we iteratively add a single link to the cycle. Specifically, sort the nodes of the ring $\{0, 1, \ldots, 2k-1\}$ in clockwise. Without loss of generality, we choose $G^{(1)} = C_n \cup e_1$, where $e_1 = (0, x)$ for some node $x : 2 \leq x \leq 2k-2$. The minimally disconnecting sets are divided into two classes: pair of links that belong either to the elementary path $P_1 = 0, 1, \ldots, x$ or $P_2 = x, x+1, \ldots, 0$. Since C_n is already 2-node connected, there is no hope to delete some of extra links e_i and obtain a minimally disconnecting set during the process for some graph $G^{(i)}$.

Mathematically, the number of minimally disconnecting sets in $G^{(1)} = C_n \cup \{(0, x)\}$ is:

$$m_2(x) = \binom{x}{2} + \binom{n-x}{2} = x^2 - nx + \frac{n^2-n}{2} \tag{4}$$

The expression for $m_2(x)$ is minimized when $x = \frac{n}{2}$. The result is Monma graph $C_n + (0, \frac{n}{2}) = M_{(\frac{n}{2}, \frac{n}{2}, 1)}$. This is not uniformly most-reliable, since the three elementary paths are not balanced. However, it represents the best single augmentation to the ring. The following steps are straightforward, using the fact that each additional link divides the classes of the minimally disconnecting sets. The first class is two pair of links from the elementary path $P_1 = 0, 1, \ldots, y_1$, where y_1 is the first node adjacent to some of the additional links e_i. The following class is $P_2 = y_1, \ldots, y_2$ is the following node adjacent to some e_i, and so on. We define the sequence $G^{(i)}$ using an analogy of *fair cake-cutting.*

Suppose that you have a cake, but the number of guests is unknown. You are the host, and guests come before midnight. Guests only require to have one piece of cake, and the only rule you have at hand is to cut the cake as minimum as possible. A cut is a division of the cake by the diameter, and it must be performed whenever a guest has no piece of cake. As soon as the first guest arrives, the cake should be cut into two identical parts (this is the first link addition). When guest number 2 arrives, the following cut is performed such that we get four identical pieces of cake (one is yours). If guest number 3 arrives, he/she already has a piece of cake, and no cut is needed. At midnight, you and your guests eat their corresponding pieces of cake. In the analogy, every cut is a link addition (that connects the farthest nodes), and the process is finite. By the previous combinatorial argument, we get the following

Theorem 1 (Fair Cake-Cutting). *The best iterative augmentation of the cycle C_n must be performed choosing the links e_i as in a finite fair cake-cutting process.*

Observe that the Cake-Cutting process is finished after $\frac{n}{2}$ steps, and the result is a special cubic graph:

Definition 4. *For every even natural n, Möbius graph M_n is constructed from the cycle C_{2n} adding n new links joining every pair of opposite nodes.*

Corollary 4. *Fare Cake-Cutting builds $M_{n/2}$.*

5. Finding Uniformly Most-Reliable Graphs

In this section we give two additional steps, studying $(n, n+i)$-graphs for $i \in \{4,5\}$. Clearly, $K_5 - e$ is the only $(n, n+4)$-graph with 5 nodes, so it is uniformly most-reliable. The cases $n \in \{6,7\}$ can be studied using exhaustive search [21]. By Proposition 1, if there exists a uniformly most-reliable graph it must be cubic (three-regular and connected). There are precisely 5 cubic graphs with 8 nodes [11]; see Figs 2-6. They were classified in [11] and systematically generated in [10]. Among $(8,12)$-graphs, the cubic graph with the highest tree-number is precisely a special Harary graph $H_{8,3}$, named Wagner graph M_4 in the memory of the author for his research on Möbius ladders. Furthermore, since t-optimal graphs must be almost regular, in this case all nodes must have the same degree; so Wagner is t-optimal.

5.1. Wagner graph.

The tree-number of Möbius graphs is [5]:

$$\tau(M_n) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2] \tag{5}$$

The tree-number in $(8,12)$-graphs is maximized in $\tau(M_4) = 392$. Therefore, the only candidate of uniformly most-reliable graph with 8 nodes is Wagner graph M_4 depicted in Figure 2. Incidentally, this graph is the product of Fair Cake-Cutting process after four augmentations. The following result can be obtained by an exhaustive search of disconnecting sets. Here we choose a combinatorial argument, since it is preparatory for the main theorem.

Lemma 1. *Wagner graph M_4 is minimizer of the coefficient m_4 among all cubic $(8,12)$-graphs.*

Proof. After the removal of 4 links, the minimum degree of the resulting graph must be $\delta \in \{0,1,2\}$. Let us denote u_i to the number of those disconnected subgraphs whose minimum degree is $\delta = i$. Therefore $m_4 = u_0 + u_1 + u_2$. Let us count and sum the three disjoint cases in order:

1. Type-0: three links incident to a fixed node and another link, $u_0 = 8 \times 9 = 72$ cases ($\delta = 0$).
2. Type-1: if we remove the four links adjacent to a fixed link we disconnect the graph. Thus, $u_1 \geq 12$.
3. Type-2: the resulting graph is 2-regular. Therefore, u_2 counts all disconnecting perfect matchings.

By inspection we see that $u_1(M_4) = 12$ and $u_2(M_4) = 2$, so $m_4(M_4) = 72+12+2 = 86$. Recall that $u_0 = 72$ and $u_1 \geq 12 = u_1(M_4)$ in all $(8,12)$-cubic graphs. In the cube $u_2(Q_3) = 3 > 2$, so m_4 is greater than in Wagner graph.

Graphs G_1, G_2 and G_3 present disconnecting sets of three non-adjacent links. There are 9 ways to pick another link. At most one of them is type-2, and clearly no-one is type-0. Therefore $u_1(G_i) \geq 12 + 8 = 20$, so $m_4(G_i) \geq 92 \geq m_4(M_4)$, for $i \in \{1,2,3\}$. □

Theorem 2. *Wagner graph is uniformly most-reliable.*

Proof. Wagner graph is t-optimal and max-λ, with $\lambda = 3$. By inspection it is super-λ, so $m_\lambda = m_3 = 8$ is minimum. Using Expression (5) we get that $m_5 = \binom{12}{5} - \tau(M_4) = 792 - 392 = 400$ is also minimum. The level of difficulty is $d(M_4) = 1$. Using Lemma 1, we know that $m_4(M_4) = 86$. It suffices to prove that

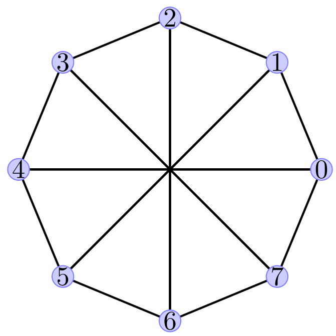

FIGURE 2. Wagner Graph M_4

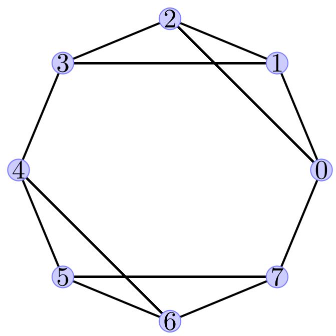

FIGURE 3. Cubic Graph G_1

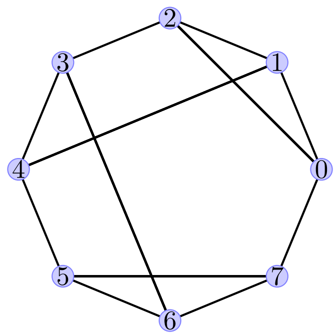

FIGURE 4. Cubic graph G_2

$m_4(H) \geq 86$ for all $(8, 12)$-graphs H.

We will study disjoint cases and use combinatorial arguments. If H has a bridge $m_4(H) \geq \binom{11}{3} > 86$. We consider bridgeless connected graphs, so in the following, $\delta(H) \geq 2$. If $\delta(H) = 2$ and we assume that $deg(v_1) = deg(v_2) = 2$ for two different nodes in H, there are two disjoint cases:

(i) v_1 and v_2 are adjacent.
(ii) v_1 and v_2 are not adjacent.

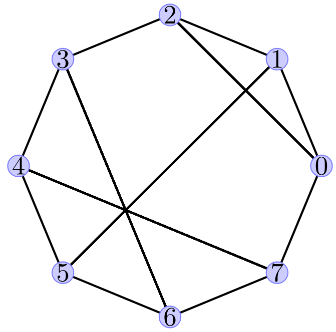

FIGURE 5. Cubic graph G_3

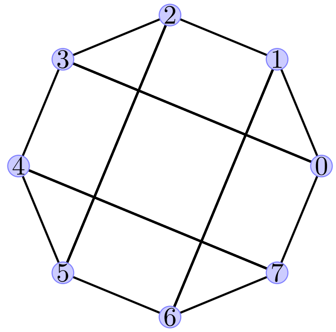

FIGURE 6. Cube Q_3

If (i), we know that $U = \{(v_1, v_2), (v_1, x), (v_2, y)\} \subseteq E(H)$ for some nodes x and y. If we pick two links from U and two from $E(H) - U$, we get disconnecting sets. Therefore, $m_4(H) \geq \binom{3}{2} \times \binom{9}{2} > 86$.

If (ii), $U = \{(v_1, x_1), (v_1, y_1), (v_2, x_2), (v_2, y_2)\} \subseteq E(H)$ for some $x, y \in V(H)$. We can pick $(v_1, x_1), (v_1, y_1)$ and two other links from $E(H) - U$: $\binom{8}{2} = 28$ cases; $(v_2, x_2), (v_2, y_2)$ and two other links from $E(H) - U$: $\binom{8}{2} = 28$ cases; or three links from U and one from $E(H) - U$: $8 \times 4 = 32$ cases. Thus $m_4(H) \geq 28 + 28 + 32 > 86$.

The remaining cases are $\delta(H) = 2$ with a single node with degree 2, or $\delta(H) \geq 3$, which must be cubic graphs. By handshaking, in the first case the degree sequence must be $(4, 3, 3, 3, 3, 3, 3, 2)$. By Erdos-Gallai characterization theorem, this sequence is graphic. We consider two disjoint disconnecting sets:

(1) The incident links of v and two additional links: $\binom{10}{2} = 45$ cases;
(2) Three incident links of a degree-3 node and another link: $6 \times 9 = 54$ cases.

Therefore, all feasible graphs with degree-sequence $(4, 3, 3, 3, 3, 3, 3, 2)$ have at least $m_4(H) \geq 45 + 54 > 86$ disconnecting sets with 4 links. The result for cubic graphs holds by Lemma 1. □

5.2. **Conjecture.** The following classes of graphs are uniformly most reliable:

- Trees among $(n, n-1)$-graphs.
- Cycles among (n, n)-graphs.
- Monma with balanced paths among $(n, n+1)$-graphs.

- K_4 and special subdivisions among $(n, n+2)$-graphs [7].
- $K_{(3,3)}$ and special subdivisions among $(n, n+3)$-graphs [26].
- Wagner graph M_4 (Theorem 3).

Trees, cycles and Monma graphs are easy graphs. They are all uniformly most-reliable graphs, and the analysis is elementary. The rationale behind the elementary subdivisions of K_4 and $K_{(3,3)}$ is identical. Indeed, the authors in both papers consider a partition into three disjoint perfect matchings, and introduce elementary subdivisions to them. All Möbius graphs accept a partition into three disjoint perfect matchings. Furthermore, $M_2 = K_4$, $M_3 = K_{(3,3)}$ and M_4 is Wagner graph. Recall that all $(n, n+2)$ and $(n, n+3)$ uniformly most-reliable graphs are obtained adding links in a balanced manner to the three disjoint perfect matchings. This promotes a conjecture:

Conjecture 1. *All uniformly most-reliable $(n, n+4)$ graphs with $n \geq 8$ are elementary subdivisions of Wagner graph.*

An optimistic prediction from Conjecture 1 is that all Möbius ladders and special subdivisions are uniformly most-reliable $(n, n+i)$ graphs. However, this generalization is false, since Petersen graph is t-optimal [17]. Curiously enough, this reinforces Donald Knuth's statement that Petersen graph serves as a counterexample to several optimistic predictions in graph theory [19].

5.3. **Petersen Graph is Uniformly Most-Reliable.** Petersen graph is the complement of the line-graph of K_5 (the reader can find alternative definitions in the book [17]). By Sachs theorem [6], its eigenvalues are 3 (simple), 1 (multiplicity 5) and -2 (multiplicity 4). Therefore, its tree-number is $\tau(P) = \frac{1}{10} \times (3-1)^5 \times (3-(-2))^4 = 2000$. By prior works in the literature it is known that Petersen is t-optimal. Therefore, Petersen graph is the only candidate to be uniformly most-reliable $(10, 15)$-graph. It is clearly super-λ with connectivity $\lambda = 3$, so $m_3 = 10$ is minimum among $(10, 15)$-graphs. From inspection we find that all the disconnecting sets with 4 links are either incident to a fixed node or fixed link, so $m_4 = 10 \times \binom{3}{3}\binom{12}{1} + 15 = 135$. Furthermore, all cubic $(10, 15)$-graphs possess the previous disconnecting sets. In order to count m_5 we observe that such disconnecting sets isolate nodes, links, 2-paths or 5-cycles, so, $m_5 = (10 \times \binom{12}{2} - 15) + 15 \times 10 + (\binom{10}{2} - 15) + 6 = 831$. From now on, we will assume that the ground graph-set is always $(10, 15)$-graphs. The following two lemmas are preparatory for the main result, and their proofs only use combinatorial arguments:

Lemma 2. *The coefficient m_4 is minimized in Petersen graph.*

Proof. The result is trivial for cubic graphs. Consider an arbitrary $(10, 15)$-graph H, and denote $m_4 = 135$ the number of disconnecting sets in Petersen graph. If H has a bridge, then $m_4(H) \geq \binom{13}{3} \geq m_4$. It suffices to prove the result when $\delta(H) = 2$. If there exists non-adjacent nodes v_1 and v_2 such that $deg(v_1) = deg(v_2) = 2$, then $m_4(H) \geq 2 \times \binom{13}{2} - 1 > m_4$. If v_1 and v_2 are adjacent nodes then $m_4(H) \geq \binom{3}{2}\binom{12}{2} > m_4$. Finally, if there is a single node v such that $deg(v) = 2$, by Handshaking Lemma the degree-sequence must be $(4, 3, 3, 3, 3, 3, 3, 3, 3, 2)$. In this case, counting the ways to disconnect a node we know that $m_4(H) \geq \binom{4}{4} + 8 \times \binom{3}{3}\binom{12}{1} + \binom{2}{2}\binom{13}{2} > m_4$. $\square$

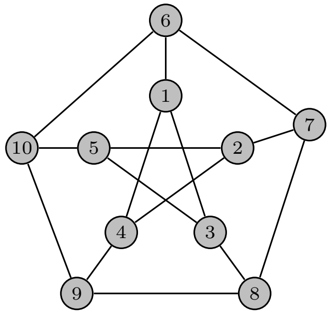

FIGURE 7. Petersen graph

Lemma 3. *The coefficient m_5 is minimized in Petersen graph among all cubic $(10, 15)$-graphs.*

Proof. □

The following result is analogous to Lemma 2:

Lemma 4. *The coefficient m_5 is minimized in Petersen graph.*

Proof. By Lemma 3 we know that the result holds for in cubic graphs. We know that $m_5 = 831$ in Petersen graph. In the following, we remark that only trivial disconnecting sets are considered for counting, unless specified otherwise. If H has a bridge, then $m_5(H) \geq \binom{14}{4} \geq m_5$. It suffices to prove the result when $\delta(H) = 2$. If $deg(v_1) = deg(v_2) = deg(v_3) = 2$ for three different nodes, we consider three disjoint and exhaustive cases:

(i) They are connected in a 2-path: $m_5(H) \geq \binom{4}{2}\binom{11}{3} + \binom{4}{3}\binom{11}{2} + \binom{4}{4}\binom{11}{1} >> m_5$

(ii) There are exactly two adjacent nodes: $m_5(H) \geq \binom{13}{3} + \binom{3}{2}\binom{12}{3} + \binom{3}{3}\binom{12}{2} - 31 \geq m_5$;

(iii) Non-adjacent nodes: $m_5(H) \geq 3 \times \binom{13}{3} - 3 \times 11 = 825$. Observe that there exists at least one node v_4 with degree 3. Adding trivial disconnecting sets related to v_4 we find $\binom{12}{2} - 3 = 63$ more sets. Therefore $m_5(H) \geq 825 + 63 = 888 > m_5$.

Assume that there are precisely two different nodes $v_1 \neq v_2$ such that $deg(v_1) = deg(v_2) = 2$. We know that $deg(v_i) = 3 + \delta_i$ for $i = 3, \ldots, 10$. By Handshaking Lemma we know that $30 = \sum_i deg(v_i)$, so $\sum_{i=3}^{10} \delta_i = 2$. Therefore, the only graphic degree-sequences with two degree-two nodes are $D_1 = (4, 4, 3, 3, 3, 3, 3, 3, 2, 2)$ and $D_2 = (5, 3, 3, 3, 3, 3, 3, 3, 2, 2)$. We consider four cases: D_1 or D_2 with adjacent or non-adjacent (A-NA) nodes v_1 and v_2:

(i) D_1 and A: $m_5(H) \geq 2 \times \binom{4}{4}\binom{11}{1} + 6 \times \binom{3}{3}\binom{12}{2} + \binom{3}{2}\binom{12}{3} + \binom{3}{3}\binom{12}{2} - 6 \times 3 = 1126 > m_5$;

(ii) D_1 and NA: $m_5(H) \geq 2 \times \binom{4}{4}\binom{11}{1} + 6 \times \binom{3}{3}\binom{12}{2} + 2 \times \binom{2}{2}\binom{13}{3} - 2 \times 5 - 11 = 969 > m_5$;

(iii) D_2 and A: $m_5(H) \geq \binom{5}{5} + 7 \times \binom{3}{3}\binom{12}{2} + \binom{3}{2}\binom{12}{3} + \binom{3}{3}\binom{12}{2} - 7 \times 3 > m_5$;

(iv) D_2 and NA: $m_5(H) \geq \binom{5}{5} + 7 \times \binom{3}{3}\binom{12}{2} + 2 \times \binom{2}{2}\binom{13}{3} - 2 \times 7 - 11 = 1010 > m_5$.

Finally, we consider the case where there exists only one degree-2 node. In this case, the degree-sequence must be $(4, 3, 3, 3, 3, 3, 3, 3, 3, 2)$. Counting trivial disconnecting sets we get that $m_5(H) \geq \binom{4}{4}\binom{11}{1} + 8 \times \binom{3}{3}\binom{12}{2} + \binom{2}{2}\binom{13}{3} - 8 = 825$. However, it does not suffice to close the proof. We must find at least 6 non-trivial disconnecting sets. Observe that there exists at least 11 links whose extremes are nodes with degree 2 or 3, so $m_5(H) \geq 825 + 11 = 836 > m_5$. $\square$

Theorem 3. *Petersen is uniformly most-reliable.*

Proof. Recall that Petersen is super-λ, so m_3 is minimized. Clearly, $m_i = 0$ for $i \in \{0,1,2\}$, and $m_i = \binom{15}{i}$ for all $(10,15)$-graphs, when $i \geq 7$. Petersen is t-optimal, thus, it minimized the coefficient m_6. Combining Lemmas 2 and 4 we know that Petersen graph minimizes simultaneously all the coefficients m_i. □

6. Conclusions and Trends for Future Work

Uniformly most-reliable graphs represent a synthesis in network reliability analysis. Finding them is a hard task not well understood. Prior works in the field try to globally minimize the coefficients of disconnecting sets. This methodology provides uniformly most-reliable $(n, n+i)$ graphs for every $i \in \{-1,0,1,2,3\}$. Here, we show the interplay between easy graphs and uniformly most-reliable graphs. We observe that Wagner graph M_4 is uniformly most-reliable. The result is suggested by an iterative augmentation of a cycle, here called Fair Cake-Cutting theorem. The paper is closed with the conjecture that $(n, n+4)$ uniformly most-reliable graphs are elementary subdivisions of Wagner graph.

There are several trends for future work. A powerful methodology to find uniformly most-reliable graphs is not known. A full characterization of t-optimal graphs is an open problem. Conjecture 1 could be studied with a parallel reasoning of Boesch [7] and Wang [26].

7. Acknowledgements

This work is partially supported by Project 395 CSIC I+D *Sistemas Binarios Estocásticos Dinámicos.*

References

1. Michael O. Ball and J. Scott Provan, *The complexity of counting cuts and of computing the probability that a graph is connected*, SIAM J. Computing **12** (1983), 777–788.
2. D. Bauer, F. Boesch, C. Suffel, and R. Van Slyke, *On the validity of a reduction of reliable network design to a graph extremal problem*, IEEE Transactions on Circuits and Systems **34** (1987), no. 12, 1579–1581.
3. D. Bauer, F. Boesch, C. Suffel, and R. Tindell, *Combinatorial optimization problems in the analysis and design of probabilistic networks*, Networks **15** (1985), no. 2, 257–271.
4. L.W. Beineke, R.J. Wilson, and O.R. Oellermann, *Topics in structural graph theory*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2012.
5. N. Biggs, *Algebraic graph theory*, Cambridge Mathematical Library, Cambridge University Press, 1993.
6. ———, *Algebraic graph theory*, Cambridge Mathematical Library, Cambridge University Press, 1993.
7. F. T. Boesch, X. Li, and C. Suffel, *On the existence of uniformly optimally reliable networks*, Networks **21** (1991), no. 2, 181–194.
8. F.T. Boesch, A. Satyanarayana, and C.L. Suffel, *A survey of some network reliability analysis and synthesis results*, Networks **54** (2009), no. 2, 99–107.
9. B. Bollobás, *Extremal graph theory*, Dover Books on Mathematics, Dover Publications, 2004.
10. Gunnar Brinkmann, Jan Goedgebeur, and Brendan D. Mckay, *Generation of Cubic graphs*, Discrete Mathematics and Theoretical Computer Science **Vol. 13 no. 2** (2011).
11. F.C Bussemake, S Cobeljic, D.M Cvetkovic, and J.J Seidel, *Cubic graphs on ≤14 vertices*, Journal of Combinatorial Theory, Series B **23** (1977), no. 2, 234 – 235.
12. E. Canale, P. Romero, G. Rubino, and X. Warnes, *Network utility problem and easy reliability polynomials*, 2016 8th International Workshop on Resilient Networks Design and Modeling (RNDM), Sept 2016, pp. 79–84.

13. Eduardo Canale, Juan Piccini, Franco Robledo, and Pablo Romero, *Diameter-constrained reliability: Complexity, factorization and exact computation in weak graphs*, Proceedings of the Latin America Networking Conference on LANC 2014 (New York, NY, USA), LANC '14, ACM, 2014, pp. 1–7.
14. Ching-Shui Cheng, *Maximizing the total number of spanning trees in a graph: Two related problems in graph theory and optimum design theory*, Journal of Combinatorial Theory, Series B **31** (1981), no. 2, 240 – 248.
15. Charles J. Colbourn, *Reliability issues in telecommunications network planning*, Telecommunications network planning, chapter 9, Kluwer Academic Publishers, 1999, pp. 135–146.
16. Frank Harary, *The maximum connectivity of a graph*, Proceedings of the National Academy of Sciences of the United States of America **48** (1962), no. 7, 1142–1146.
17. D.A. Holton and J. Sheehan, *The petersen graph*, Australian Mathematical Society Lecture Series, Cambridge University Press, 1993.
18. G. Kirchoff, *Über die auflösung der gleichungen, auf welche man bei der untersuchung der linearen verteilung galvanischer ströme geführt wird*, Ann. Phys. Chem. **72** (1847), 497–508.
19. D.E. Knuth, *The art of computer programming: Introduction to combinatiorial algorithms and boolean functions*, Addison-Wesley series in Computer Science and Information Proceedings, Addison-Wesley, 2008.
20. J. D. Leggett and S. D. Bedrosian, *On networks with the maximum numbers of trees*, Proc. of Eighth Midwest Symposium on Circuit Theory, June 1965, pp. 1–8.
21. Behrang Mahjani, *Exploring connectivity of random subgraphs of a graph*, Ph.D. thesis, Chalmers University of Technology, Göteburg, Sweden, 2010.
22. ClydeL. Monma, BethSpellman Munson, and WilliamR. Pulleyblank, *Minimum-weight two-connected spanning networks*, Mathematical Programming **46** (1990), no. 1-3, 153–171 (English).
23. Wendy Myrvold, Kim H. Cheung, Lavon B. Page, and Jo Ellen Perry, *Uniformly-most reliable networks do not always exist*, Networks **21** (1991), no. 4, 417–419.
24. L. Petingi, F. Boesch, and C. Suffel, *On the characterization of graphs with maximum number of spanning trees*, Discrete Mathematics **179** (1998), no. 1, 155 – 166.
25. F. Robledo, P. Romero, and P. Sartor, *A novel interpolation technique to address the edge-reliability problem*, 2013 5th International Congress on Ultra Modern Telecommunications and Control Systems and Workshops (ICUMT), Sept 2013, pp. 187–192.
26. Guifang Wang, *A proof of Boesch's conjecture*, Networks **24** (1994), no. 5, 277–284.

Facultad de Ingeniería. Universidad de la República, Uruguay
E-mail address: promero@fing.edu.uy