EQUIVARIANT ALGEBRAIC KK-THEORY

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1. Kasparov's KK-theory

Kasparov's KK-theory is the major tool in *noncommutative topology*, [10]. The KK-theory of separable C^*-algebras is a common generalization both of topological K-homology and tolopological K-theory as an *additive bivariant functor* Let A and B separable C^*-algebras then a *group* $KK(A,B)$ is defined such that

$$KK_*(\mathbb{C}, B) \simeq K_*^{top}(B) \qquad KK^*(A, \mathbb{C}) \simeq K^*_{hom}(A).$$

An important property of KK-theory is the so-called *Kasparov product*,

$$KK(A,B) \times KK(B,C) \to KK(A,C)$$

which is bilinear with respect to the additive group structures. The Kasparov groups $KK(A,B)$ for $A, B \in C^*$-Alg form a morphisms sets $A \to B$ of a *category* KK. The composition in KK is given by the Kasparov product and the category KK admits a *triangulated category* structure.

There is a *canonical functor* $k : C^*\text{-Alg} \to KK$ that acts identically on objects and every *-homomorphism $f : A \to B$ is represented by an element $[f] \in KK(A,B)$. The functor $k : C^*\text{-Alg} \to KK$

- ... is homotopy invariant: $f_0 \sim f_1$ implies $k(f_0) = k(f_1)$.
- ... is C^*-stable: any corner embedding $A \to A \otimes \mathcal{K}(\ell^2\mathbb{N})$ induces an isomorphism $k(A) = k(A \otimes \mathcal{K}(\ell^2\mathbb{N}))$.
- ... is split-exact: for every split-extension $I \xrightarrow{f} A \xrightarrow{g} A/I$ (i.e. there exists a *-homomorpfhism $s : A/I \to A$ such that $g \circ s = \mathrm{id}$) then $k(I) \xrightarrow{k(f)} k(A) \xrightarrow{k(g)} k(A/I)$ is part of a distinguished triangle.

The functor $k : C^*\text{-Alg} \to KK$ is the *universal* homotopy invariant, C^*-stable and split exact functor. Main authors who worked in the previous results: J. Cuntz, N. Higson, G. Kasparov, R. Meyer.

2. Algebraic kk-theory

Algebraic kk-theory was introduced by G. Cortiñas and A. Thom in order to show how methods from K-theory of operator algebras can be applied in completely algebraic setting. Let ℓ a commutative ring with unit and Alg the category of ℓ-algebras (with or without unit).

Kasparov's KK-theory $\leftrightarrow$ Algebraic kk-theory

Kasparov's KK-theory

[10]

bivariant K-theory on C^*-Alg

$k : C^*\text{-Alg} \to KK$

k is stable w.r.t. compact operators

$$A \simeq_{KK} A \otimes \mathcal{K}(\ell^2(\mathbb{N}))$$

k is continuous homotopy invariant

$$\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[dr, "H"'] & B \\
& C([0,1],B) \arrow[u, bend left, "\mathrm{ev}_0"] \arrow[u, bend right, "\mathrm{ev}_1"']
\end{tikzcd}$$

$$B \simeq_{KK} C([0,1],B)$$

k is split exact

k is universal for these properties

$$\boxed{KK_*(\mathbb{C}, A) \simeq K_*^{top}(A)}$$

Algebraic kk-theory

[2]

bivariant K-theory on Alg

$j : \mathrm{Alg} \to \mathfrak{K}\mathfrak{K}$

j is stable w.r.t. matrices

$$A \simeq_{\mathfrak{K}\mathfrak{K}} M_\infty(A) = \bigcup_{n\in\mathbb{N}} M_n(A)$$

j is polynomial homotopy invariant

$$\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[dr, "H"'] & B \\
& B[t] \arrow[u, bend left, "\mathrm{ev}_0"] \arrow[u, bend right, "\mathrm{ev}_1"']
\end{tikzcd}$$

$$B \simeq_{\mathfrak{K}\mathfrak{K}} B[t]$$

j is excisive

j is universal for these properties

$$\boxed{kk_*(\ell, A) \simeq \mathrm{KH}_*(A)}$$

KH is Weibel's homotopy K-theory defined in [19].

Theorem 2.1. [2] *The functor $j : \mathrm{Alg} \to \mathfrak{K}\mathfrak{K}$ is an excisive, homotopy invariant, and M_∞-stable functor and it is the universal functor for these properties.*

Let $\mathcal{X}$ be a infinity set. Consider

$$M_\mathcal{X} := \{a : \mathcal{X} \times \mathcal{X} \to \ell : \mathrm{sopp}(a) < \infty\}.$$

Let A be an algebra, then $M_\mathcal{X} A := M_\mathcal{X} \otimes_\ell A$.

Theorem 2.2. [13] *The functor $j : \mathrm{Alg} \to \mathfrak{R}_\mathcal{X}$ is an excisive, homotopy invariant, and $M_\mathcal{X}$-stable functor and it is the universal functor for these properties.*

If $\mathcal{X} = \mathbb{N}$ both theorems are the same.

3. Equivariant algebraic kk-theory

We introduce in [5] an algebraic bivariant K-theory for the category of G-algebras where G is a group.

Equivariant Kasparov's KK-theory $\leftrightarrow$ Equivariant algebraic kk-theory

Equivariant Kasparov's KK-theory

[10]

bivariant K-theory on G-C^*-Alg

$k : G\text{-}C^*\text{-Alg} \to KK^G$

k is stable w.r.t. compact operators

$$A \simeq_{KK^G} A \otimes \mathcal{K}(\ell^2(G \times \mathbb{N}))$$

k is continuous homotopy invariant

$$B \simeq_{KK^G} C([0,1],B)$$

k is split exact

k is universal for these properties

G compact

$$\boxed{KK_*^G(\mathbb{C}, A) \simeq K_*^{top}(A \rtimes G)}$$

Equivariant algebraic kk-theory

[5]

bivariant K-theory on G-Alg

$j : G\text{-Alg} \to \mathfrak{K}\mathfrak{K}^G$

j is G-stable

$$A \simeq_{\mathfrak{K}\mathfrak{K}^G} M_\infty M_G(A)$$

j is polynomial homotopy invariant

$$B \simeq_{\mathfrak{K}\mathfrak{K}^G} B[t]$$

j is excisive

j is universal for these properties

G finite and $\frac{1}{|G|} \in \ell$

$$\boxed{kk_*^G(\ell, A) \simeq KH_*(A \rtimes G)}$$

Let A be a G-algebra and

$$M_G := \{a : G \times G \to \ell : \mathrm{sopp}(a) < \infty\}.$$

Consider in $M_G \otimes A$ the following action of G

$$g \cdot (e_{s,t} \otimes a) = e_{gs,gt} \otimes g \cdot a$$

A *G-stable* functor identifies any G-algebra A with $M_G \otimes A$.

Theorem 3.1 ([5])**.** *The functor $j : G\text{-Alg} \to \mathfrak{K}\mathfrak{K}^G$ is an excisive, homotopy invariant, and G-stable functor and it is the universal functor for these properties.*

3.1. **Green-Julg Theorem.** The functors

$$\begin{tikzcd} G\text{-Alg} \arrow[r, bend left, "\rtimes"] & \mathrm{Alg} \arrow[l, bend left, "\tau"] \end{tikzcd}$$

$$A \rtimes G = A \otimes \ell G \qquad (a \ltimes g)(b \ltimes h) = a[g \cdot b] \ltimes gh$$

can be extended to

$$\begin{tikzcd} G\text{-Alg} \arrow[r, bend left, "\rtimes"] \arrow[d, "j^G"'] & \mathrm{Alg} \arrow[l, bend left, "\tau"] \arrow[d, "j"] & \text{not adjoint functors} \arrow[l] \\ \mathfrak{K}\mathfrak{K}^G \arrow[r, bend left, "\rtimes"] & \mathfrak{K}\mathfrak{K} \arrow[l, bend left, "\tau"] & \text{adjoint functors if } G \text{ is finite and } \frac{1}{|G|} \in \ell \arrow[l] \end{tikzcd}$$

Theorem 3.2. [5] *Let G be a finite group of n elements such that $\frac{1}{n} \in \ell$. Let A be a G-algebra and B an algebra. There is an isomorphism*

$$\psi_{GJ} : kk^G(B^\tau, A) \to kk(B, A \rtimes G)$$

Corollary 3.3. [5] *Let G be a finite group of n elementrs such that $\frac{1}{n} \in \ell$. Then*

$$kk^G(\ell, A) \simeq \mathrm{KH}(A \rtimes G) \qquad kk^G(\ell, \ell) \simeq \mathrm{KH}(\ell G)$$

3.2. **Adjointness between Ind_H^G and Res_G^H.** Let H be a subgroup of G and A an H-algebra. Define

- $A^{(G)} = \{\alpha : G \to A : \alpha \text{ is a function with finite support}\}$
- $\mathrm{Ind}_H^G(A) = \{\alpha \in A^{(G)} : \alpha(s) = h \cdot \alpha(sh) \quad \forall h \in H, s \in G\}$
- $(g \cdot \alpha)(s) = \alpha(g^{-1}s)$

The functors

$$\begin{tikzcd} G\text{-Alg} \arrow[r, bend left, "\mathrm{Res}_G^H"] & H\text{-Alg} \arrow[l, bend left, "\mathrm{Ind}_H^G"] & \mathrm{Ind}_H^G \text{ is NOT left adjoint to } \mathrm{Res}_G^H \arrow[l] \end{tikzcd}$$

can be extended to

$$\begin{tikzcd} \mathfrak{K}\mathfrak{K}^G \arrow[r, bend left, "\mathrm{Res}_G^H"] & \mathfrak{K}\mathfrak{K}^H \arrow[l, bend left, "\mathrm{Ind}_H^G"] & \mathrm{Ind}_H^G \text{ is a left adjoint to } \mathrm{Res}_G^H \arrow[l] \end{tikzcd}$$

Theorem 3.4. [5] *Let G be a group, H a subgroup of G, B an H-algebra and A a G-algebra. Then there is an isomorphism*

$$\psi_{IR} : kk^G(\mathrm{Ind}_H^G(B), A) \to kk^H(B, \mathrm{Res}_G^H(A))$$

Corollary 3.5.

- $kk^G(\ell^{(G/H)}, A) \simeq kk^H(\ell, \mathrm{Res}_G^H(A))$.
- *If H is finite then* $kk^G(\ell^{(G/H)}, A) \simeq \mathrm{KH}(A \rtimes H)$.
- $kk^G(\ell^{(G)}, A) \simeq \mathrm{KH}(A)$.

3.3. **Baaj-Skandalis duality.** A *G-graduation* on an algebra A is a decomposition on submodules

$$A = \bigoplus_{s \in G} A_s \quad A_s A_t \subseteq A_{st} \quad \forall s, t \in G$$

The functors

$$\begin{tikzcd} G\text{-Alg} \arrow[r, bend left, "\rtimes"] & G_{gr}\text{-Alg} \arrow[l, bend left, "\hat{\rtimes}"] \end{tikzcd}$$

can be extended to

$$\begin{tikzcd} G\text{-Alg} \arrow[r, bend left, "\rtimes"] \arrow[d, "j^G"'] & G_{gr}\text{-Alg} \arrow[l, bend left, "\hat{\rtimes}"] \arrow[d, "\hat{j}^G"] & \text{not an equivalence} \arrow[l] \\ \mathfrak{KK}^G \arrow[r, bend left, "\rtimes"] & \hat{\mathfrak{KK}}^G \arrow[l, bend left, "\hat{\rtimes}"] & \text{an equivalence} \arrow[l] \end{tikzcd}$$

4. Algebraic quantum kk-theory

4.1. **Van Daele's algebraic quantum groups.** Let $\ell = \mathbb{C}$ and $(\mathcal{G}, \Delta, \varphi)$ an algebraic quantum group. That means, $(\mathcal{G}, \Delta)$ is a *multiplier Hopf algebra*:

- $\mathcal{G}$ associative algebra over $\mathbb{C}$ with non-degenerate product
- $M(\mathcal{G})$ multiplier algebra of $\mathcal{G}$: $(\rho_1, \rho_2) \in M(\mathcal{G})$ if
  - $\rho_i : \mathcal{G} \to \mathcal{G}$ is a linear map $(i = 1, 2)$
  - $\rho_1(hk) = \rho_1(h)k \quad \rho_2(hk) = h\rho_2(k) \quad \rho_2(h)k = h\rho_1(k) \quad \forall h, k \in \mathcal{G}$
  - $(\rho_1, \rho_2)(\tilde{\rho}_1, \tilde{\rho}_2) = (\rho_1\tilde{\rho}_1, \tilde{\rho}_2\rho_2)$
- An homomorphism $\Delta : \mathcal{G} \to M(\mathcal{G} \otimes \mathcal{G})$ is a comultiplication if
  - $\Delta(h)(1 \otimes k) \in \mathcal{G} \otimes \mathcal{G} \qquad (h \otimes 1)\Delta(k) \in \mathcal{G} \otimes \mathcal{G} \quad \forall h, k \in \mathcal{G}$
  - The coassociativity property is satisfied:

$$(h \otimes 1 \otimes 1)(\Delta \otimes \mathrm{id}_{\mathcal{G}})(\Delta(k)(1 \otimes r)) = (\mathrm{id}_{\mathcal{G}} \otimes \Delta)((h \otimes 1)\Delta(k))(1 \otimes 1 \otimes r) \qquad \forall h, k, r \in \mathcal{G}$$

- The following maps are bijective:

$$T_i : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G} \otimes \mathcal{G} \qquad T_1(h \otimes k) = \Delta(h)(1 \otimes k) \qquad T_2(h \otimes k) = (h \otimes 1)\Delta(k)$$

Proposition 4.1. *If $(\mathcal{G}, \Delta)$ is a multiplier Hopf algebra there is a unique homomorphism $\epsilon : \mathcal{G} \to \mathbb{C}$, called counit, such that*

$$(\epsilon \otimes \mathrm{id}_{\mathcal{G}})(\Delta(h)(1 \otimes k)) = hk \qquad (\mathrm{id}_{\mathcal{G}} \otimes \epsilon)((h \otimes 1)\Delta(k)) = hk \qquad \forall h, k \in \mathcal{G}.$$

There is also a unique anti-homomorphism $S : \mathcal{G} \to M(\mathcal{G})$, called antipode, such that

$$m(S \otimes \mathrm{id}_{\mathcal{G}})(\Delta(h)(1 \otimes k)) = \epsilon(h)k \qquad m(\mathrm{id}_{\mathcal{G}} \otimes S)((h \otimes 1)\Delta(k)) = \epsilon(k)h \qquad \forall h, k \in \mathcal{G}$$

here m is the multiplication map.

$(\mathcal{G}, \Delta)$ is a *regular multiplier Hopf algebra* if $S(\mathcal{G}) \subseteq \mathcal{G}$ and S is invertible. There is a natural embedding $\iota_{\mathcal{G}} : \mathcal{G} \to M(\mathcal{G})$ which is an homomorphism

$$h \mapsto (L_h, R_h) \qquad L_h(k) = hk \qquad R_h(k) = kh$$

Moreover $\rho h \in \mathcal{G}$ and $h\rho \in \mathcal{G}$ for all $h \in \mathcal{G}$ and $\rho \in M(\mathcal{G})$,

$$\rho h = (L_{\rho_1(h)}, R_{\rho_1(h)}) \qquad h\rho = (L_{\rho_2(h)}, R_{\rho_2(h)})$$

We write $\rho h = \rho_1(h)$ and $h\rho = \rho_2(h)$.

A *right invariant functional* on $\mathcal{G}$ is a non-zero linear map $\psi : \mathcal{G} \to \mathbb{C}$ such that

$$(\psi \otimes \mathrm{id}_{\mathcal{G}})\Delta(h) = \psi(h)1$$

Here $(\psi \otimes \mathrm{id}_{\mathcal{G}})\Delta(h)$ denotes the element $\rho \in M(\mathcal{G})$ such that

$$\rho k = (\psi \otimes \mathrm{id}_{\mathcal{G}})(\Delta(h)(1 \otimes k)) \qquad k\rho = (\psi \otimes \mathrm{id}_{\mathcal{G}})((1 \otimes k)\Delta(h))$$

Similarly, a *left invariant functional* on $\mathcal{G}$ is a non-zero linear map $\varphi : \mathcal{G} \to \mathbb{C}$ such that

$$(\mathrm{id}_{\mathcal{G}} \otimes \varphi)\Delta(h) = \varphi(h)1.$$

Invariant functionals do not always exist. If φ is a left invariant functional on $\mathcal{G}$ then it is unique up to scalar multiplication and $\psi = \varphi \circ S$ is a right invariant functional.

$$\boxed{\text{ALGEBRAIC QUANTUM GROUP}}$$
$$=$$
$$\boxed{\text{REGULAR MULTIPLIER HOPF ALGEBRA WITH INVARIANTS}}$$

The dual of $(\mathcal{G}, \Delta)$ is $(\hat{\mathcal{G}}, \hat{\Delta})$:

- The elements of $\hat{\mathcal{G}}$ are the linear functionals of the form $\varphi(h\cdot)$

$$\hat{\mathcal{G}} = \{\xi_h : \mathcal{G} \to \mathbb{C} : \xi_h(x) = \varphi(hx)\}$$

  The elements of $\hat{\mathcal{G}}$ can also be written as $\varphi(\cdot h)$, $\psi(h\cdot)$, $\psi(\cdot h)$.
- The product on $\hat{\mathcal{G}}$ is defined as follows

$$(\xi_h \cdot \xi_k)(x) = (\varphi \otimes \varphi)(\Delta(x)(h \otimes k))$$

- The coproduct $\hat{\Delta} : \hat{\mathcal{G}} \to M(\hat{\mathcal{G}} \otimes \hat{\mathcal{G}})$ is defined by defining the elements $\hat{\Delta}(\xi_1)(1 \otimes \xi_2)$ and $(\xi_1 \otimes 1)\hat{\Delta}(\xi_2)$ in $\hat{\mathcal{G}} \otimes \hat{\mathcal{G}}$ as follows

$$((\xi_1 \otimes 1)\hat{\Delta}(\xi_2))(h \otimes k) = (\xi_1 \otimes \xi_2)(\Delta(h)(1 \otimes k))$$

$$(\hat{\Delta}(\xi_1)(1 \otimes \xi_2))(h \otimes k) = (\xi_1 \otimes \xi_2)((h \otimes 1)(\Delta(k))$$

- $(\hat{\mathcal{G}}, \hat{\Delta})$ is isomorphic to $(\mathcal{G}, \Delta)$ as algebraic quantum group

4.2. **Examples.**

- $\mathcal{G} = \mathbb{C}G$ with the usual Hopf algebra structure.

$$\varphi = \psi = \chi_e : G \to \mathbb{C} \quad \chi_e(h) = \begin{cases} 1 & e = h \\ 0 & e \neq h \end{cases}$$

  $\boxed{\mathcal{G} \text{ is compact type}}$ because $1 \in \mathcal{G}$.

- $\mathcal{G} = \mathbb{C}\hat{G} = \{\sum_{g\in G} a_g\chi_g : a_g \in \mathbb{C}\ a_g \neq 0 \text{ for a finite amount of } g\}$
$$\chi_g\chi_h = \left\{ \begin{array}{ll} \chi_g & g = h \\ 0 & g \neq h \end{array} \right.$$
$$\Delta : \mathcal{G} \to M(\mathcal{G}\otimes\mathcal{G}) \quad \Delta(\chi_g) = \sum_{t\in G} \chi_{gt^{-1}} \otimes \chi_t$$
The integral is $\varphi = \psi : \mathcal{G} \to \mathbb{C} \qquad \varphi(\chi_h) = \psi(\chi_h) = 1$
$\boxed{\mathcal{G} \text{ is discrete type}}$ because exists $k \in \mathcal{G} \quad xk = \epsilon(x)k$.
- $\mathcal{G} = \mathcal{H}$ a finite dimensional Hopf algebra. $\boxed{\mathcal{G} \text{ is compact and discrete}}$

Let $(\mathcal{G}, \Delta)$ be an algebraic quantum group and A be a $\mathcal{G}$-module algebra.

$$\boxed{\hat{\mathcal{A}}(\mathcal{G}) := \mathcal{G} \underset{\text{ev}}{\otimes} \hat{\mathcal{G}}} \qquad (g\otimes f)(\tilde{g}\otimes\tilde{f}) = gf(\tilde{g})\otimes\tilde{f}$$

$$t\cdot(g\otimes f) = \sum t_{(1)}\cdot g\otimes t_{(2)}\cdot f$$

$$(t\cdot f)(g) = f(S(t)g)$$

$$\boxed{\hat{\mathcal{A}}(\mathcal{G})\otimes A} \qquad (g\otimes f\otimes a)(\tilde{g}\otimes\tilde{f}\otimes\tilde{a}) = gf(\tilde{g})\otimes\tilde{f}\otimes a\tilde{a}$$

$$t\cdot(g\otimes f\otimes a) = \sum t_{(1)}\cdot g\otimes t_{(3)}\cdot f\otimes t_{(2)}\cdot a$$

A functor $F : \mathcal{G}\text{-Alg} \to \mathcal{D}$ is $\mathcal{G}$-stable if $F(\iota_1)$ and $F(\iota_2)$ are isomorphism where

$$\iota_2 : A \to \begin{pmatrix} \hat{\mathcal{A}}(\mathcal{G})\otimes A & 0 \\ 0 & A \end{pmatrix} \leftarrow \hat{\mathcal{A}}(\mathcal{G})\otimes A : \iota_1$$

are corner inclusions.

5. Algebraic quantum kk-theory

Theorem 5.1. *Let $\mathcal{X}$ be a set such that* $\operatorname{card}(\mathcal{X}) = \mathbb{N}\times\dim_{\mathbb{C}}(\mathcal{G})$. *Let $F : \mathcal{G}$-Alg $\to \mathcal{D}$ be a $M_{\mathcal{X}}$-stable functor. The functor*

$$\hat{F} : \mathcal{G}\text{-Alg} \to \mathcal{D} \qquad A \mapsto F(\hat{\mathcal{A}}(\mathcal{G})\otimes A)$$

is $\mathcal{G}$-stable.

Theorem 5.2. *Theorem The functor $j^{\mathcal{G}} : \mathcal{G}\text{-Alg} \to \mathfrak{KK}^{\mathcal{G}}$ is an excisive, homotopy invariant, and $\mathcal{G}$-stable functor. Moreover, it is the universal functor for these properties.*

6. Adjointness theorems in algebraic quantum kk-theory

6.1. Green-Julg theorem.

$$A\#\mathcal{H} = A\otimes\mathcal{H} \qquad (a\#h)(b\#k) = \sum a(h_{(1)}\cdot b)\#h_{(2)}k$$

Theorem 6.1. *Let $\mathcal{H}$ be a semisimple Hopf algebra and A be an $\mathcal{H}$-module algebra then*

$$kk^{\mathcal{H}}(\mathbb{C}, A) \simeq \mathrm{KH}(A\#\mathcal{H})$$

6.1. **Green-Julg theorem.** Let $\mathcal{G}$ be an algebraic quantum group.

- $\hat{\mathcal{G}}$ is a $\mathcal{G}$-module: $(g \rightharpoonup f)(k) = f(kg)$
- $\mathcal{G}$ is a $\hat{\mathcal{G}}$-module: $f \rightharpoonup g = \sum f(g_{(2)})g_{(1)}$

Theorem 6.2. *(B. Drabant, A. Van Daele, Y. Zhang) Let A be a $\mathcal{G}$-module algebra then*

$$(A\#\mathcal{G})\#\hat{\mathcal{G}} \simeq \hat{\mathcal{A}}(\mathcal{G}) \otimes A$$

Theorem 6.3. *The functors $\#\mathcal{G} : \mathfrak{KK}^{\mathcal{G}} \to \mathfrak{KK}^{\hat{\mathcal{G}}}$ $\quad \#\hat{\mathcal{G}} : \mathfrak{KK}^{\hat{\mathcal{G}}} \to \mathfrak{KK}^{\mathcal{G}}$ are equivalences and*

$$kk^{\mathcal{G}}(A, B) \simeq kk^{\hat{\mathcal{G}}}(A\#\mathcal{G}, B\#\mathcal{G})$$

where A, B are $\mathcal{G}$-module algebras.

7. Conclusion

	G group	$\mathcal{G}$ algebraic quantum group
equivariant $\mathfrak{KK}$	$\mathfrak{KK}^G$	$\mathfrak{KK}^{\mathcal{G}}$
stability	$\boxed{B \simeq_{\mathfrak{KK}^G} M_G B}$	$\boxed{B \simeq_{\mathfrak{KK}^{\mathcal{G}}} \hat{\mathcal{A}}(\mathcal{G}) \otimes B}$
Green-Julg Theorem	$\mathfrak{KK}^G \underset{\tau}{\overset{\rtimes}{\rightleftarrows}} \mathfrak{KK}$ adjoints functors G is finite, $\frac{1}{\lvert G\rvert} \in \ell$ $\boxed{kk_*^G(\ell, A) \simeq KH_*(A \rtimes G)}$	$\mathfrak{KK}^{\mathcal{G}} \underset{\tau}{\overset{\#}{\rightleftarrows}} \mathfrak{KK}$ adjoint functors if $\mathcal{G} = \mathcal{H}$ semisimple Hopf algebra $\boxed{kk_*^{\mathcal{H}}(\ell, A) \simeq KH_*(A\#\mathcal{H})}$
Ind-Res	$\mathfrak{KK}^G \underset{\mathrm{Ind}_H^G}{\overset{\mathrm{Res}_G^H}{\rightleftarrows}} \mathfrak{KK}^H$ adjoint functors	$\boxed{?}$
Imprimitivity	$\boxed{\mathrm{Ind}_H^G(B) \rtimes G \simeq_{\mathfrak{KK}} B \rtimes H}$	$\boxed{?}$
Baaj-Skandalis duality	$\mathfrak{KK}^G \underset{\hat{\rtimes}}{\overset{\rtimes}{\rightleftarrows}} \hat{\mathfrak{KK}}^G$ equivalences	$\mathfrak{KK}^{\mathcal{G}} \underset{\#}{\overset{\#}{\rightleftarrows}} \mathfrak{KK}^{\hat{\mathcal{G}}}$ equivalences

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